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Nature's Natural Numbers: Relativistic Quantum Theory over the Ring of Complex Quaternions

JAMES D. EDMONDS, Jr.

Department of Physics, California Western College, San Diego, California 92106

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Abstract

The 2×2 complex matrix formulation of relativity and the two-component spin-$\frac{1}{2}$ formalism are merged with the complex quaternion algebra to yield a concise, manifestly covariant formalism of relativistic quantum mechanics. Along with reproducing all the old results of quantum theory, this complex quaternion formulation extends naturally the concept of scalar mass by adding to it orientation- and velocity-dependent parts giving a hyper-mass. The hyper-mass spin-$\frac{1}{2}$ equation, with the scalar part of the mass set equal to zero, gives a subtle variation on the two-component neutrino theory with very unsubtle consequences, such as an invariant mass parameter which could distinguish ν_e and ν_μ and elimination of the superposition principle.

1. *Introduction*

The real quaternion field is well known to most mathematicians from group theory, but it is not so generally known in the physics community. Most physicists are very familiar with the algebra of Pauli matrices, which is very similar to that of quaternions as we show in the next section. The relative merits of the quaternion formulation, due to Hamilton and supported by Tait, and the newer vector analysis of Gibbs and Heaviside was hotly disputed (Bork, 1966) from about 1880 to 1900. The vector enthusiasts won out, obviously; but as we shall see, Hamilton's quaternions are well suited for relativistic problems. Einstein's four-dimensional space-time was not introduced until 1905, when the vector formalism was already well established. The quaternion formalism is directly applicable to three space and one time dimension, and to spin-$\frac{1}{2}$. To a mathematician this is a severe limitation, but for the real world at a fundamental level these conditions appear to dominate. In any case, Hestenes (1966, 1968, 1971) and others have shown how to generalize this approach, using Clifford's algebra to obtain a 'multi-vector' formulation applicable to any dimensional space.

In recent years there has been considerable interest in finding useful generalizations of quantum mechanics (Gürsey, 1956a, 1958; Misra, 1960; Phipps, 1960; Pais, 1961; Tiomno, 1963; Jordan, 1968; Penney, 1968; Biedenharn *et al.*, 1971), many of these involving quaternions (Klein, 1911; Lanczos, 1929; Birkhoff & von Neumann, 1936; Conway, 1937; Brand, 1947; Yang & Mills, 1954; Gürsey, 1956a, b, c; Schremp, 1959; Kaneno, 1960; Allcock, 1961; Dyson, 1962; Finkelstein *et al.*, 1962a, b; Winans, 1962; Emch, 1963; Finkelstein *et al.*, 1963; Jenč, 1966; Natarajan & Viswanath, 1967). Because of the excellent agreement between standard non-relativistic quantum mechanics and experiment, we should expect that any useful generalization must involve the relativistic realm where the existing theories have obvious shortcomings.

The 2×2 complex matrices provide the bridge between conventional tensor analysis and quaternion analysis. Recently, Sachs has shown how to formulate Einstein's general relativity using 2×2 complex matrices (Newman & Penrose, 1962; Rastall, 1964; Sachs, 1968, 1970). Wightman, MacFarlane, and others have shown how to formulate Lorentz transformations using 2×2 complex matrices [the group $SL(2, C)$] (MacFarlane, 1962; Naimark, 1964; Streater & Wightman, 1961; Carruthers, 1966). MacFarlane's extensive treatment contains the equivalent of the Lorentz transformation results summarized in this paper, though the notation is different. An elegant treatment of Dirac theory in terms of a two-component wave equation has been given by Brown (1962), in which it is shown that all the usual results can also be obtained in this formalism.

The major portion of the present article outlines the basic formalism of relativistic quantum mechanics using the complex quaternion ring notation. This notation, with the hyperconjugation operation, is very simple and streamlined. The majority of our formulation is equivalent to that contained in MacFarlane's and Brown's papers.

A new and potentially important concept emerges, however, from the hypercomplex number approach. By generalizing from a complex number algebra to a complex quaternion algebra, we are naturally led to explore the possibility that observables should be generalized from real numbers to some kind of hypercomplex numbers such that in the non-relativistic limit they reduce back to real numbers. This pattern of generalization has repeated itself many times in the advance of physics.

We show in Section 6 that a conserved probability current can easily be obtained only if $\hbar$ is not generalized and remains a real scalar. The generalization of mass, by adding space-like parts to the real scalar part, seems to be compatible with all the usual requirements of quantum mechanics, except that the superposition principle is eliminated and other observables such as energy become hypercomplex. Particles with zero scalar mass but non-zero space-like mass could provide a way of describing the different kinds of neutrinos. The corresponding wave equation is similar to that of the two-component neutrino theory.

It is our point of view that the proven importance of the Pauli matrices and the $SU(3)$ group, along with the dimensionality of space-time, strongly suggest that the ring of complex quaternions constitutes nature's natural number system. If the hyper-mass turns out to be helpful in understanding neutrinos, then we will have convincing proof.

2. *Space-Time Algebra*

2.1. *Flat Space-Time*

In the hypercomplex number formulation of space-time, we represent a space-time event, with rectangular coordinates (x,y,z,t), by the number

$$x \equiv x^\mu e_\mu \tag{2.1.1}$$

where $\{x^\mu, \mu = 0,1,2,3,\} = \{ct,x,y,z\}$ and the independent elements e_μ are isomorphic to the Pauli matrices:

$$\begin{aligned} &e_0 e_0 \equiv e_0; \qquad e_k e_0 \equiv e_0 e_k \equiv e_k, \qquad k = 1,2,3; \qquad e_k e_k = e_0 \\ &e_1 e_2 \equiv ie_3, \qquad i \equiv \sqrt{-1} \text{ (cycl. perm.)} \end{aligned} \tag{2.1.2}$$

and

$$e_j e_k \equiv -e_k e_j, \qquad k \neq j, k \neq 0, j \neq 0$$

We will find it useful to define eight component hypercomplex numbers

$$q \equiv (q_R{}^\mu + iq_I{}^\mu) e_\mu \tag{2.1.3}$$

(isomorphic to the ring of complex quaternions) and, in addition to the usual complex conjugate q^*, the hyperconjugate $q^\ddagger$, where

$$\begin{aligned} &q^* \equiv (q_R{}^\mu - iq_I{}^\mu) e_\mu, \qquad q^\ddagger \equiv ({}_R{}^\mu + iq_I{}^\mu) e_\mu{}^\ddagger \\ &e_0{}^\ddagger \equiv e_0, \qquad \text{and} \qquad e_k{}^\ddagger \equiv -e_k, k = 1,2,3 \end{aligned} \tag{2.1.4}$$

We can readily show that, given two hypercomplex numbers p and q,

$$(pq)^\ddagger = q^\ddagger p^\ddagger, \qquad (pq)^* = q^* p^*, \qquad (pq)^{\ddagger *} = p^{\ddagger *} q^{\ddagger *} \tag{2.1.5}$$

and $(p^\ddagger)^* = (p^*)^\ddagger$.

Hypercomplex numbers q for which $q^\ddagger = q^*$ are isomorphic to the field of 'real' quaternions; $q^\ddagger = q$ means q is a complex number; $q^* = q = q^\ddagger$ gives a real number; $q^\ddagger = -q$ means that $q^0 = 0$, and will be called a space-like number.

The flat space-time metric is given by

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu e_0, \tag{2.1.6}$$

where

$$g_{\mu\nu} \equiv \tfrac{1}{4}(e_\mu{}^\ddagger e_\nu + e_\mu e_\nu{}^\ddagger + e_\nu{}^\ddagger e_\mu + e_\nu e_\mu{}^\ddagger) \tag{2.1.7}$$

This gives the usual signature $+---$. We define the inner product of two events x and y by

$$(x|y) \equiv g_{\mu\nu} x^\mu y^\nu e_0 = \tfrac{1}{4}(x^\ddagger y + xy^\ddagger + y^\ddagger x + yx^\ddagger) \tag{2.1.8}$$

Proper Lorentz transformations can now be defined by

$$x = x^{\mu} e_{\mu} \to x^{\mu\prime} e_{\mu} = \mathscr{L}^{*} x \mathscr{L} \tag{2.1.9}$$

where

$$\mathscr{L} \equiv \mathscr{L}^{\mu} e_{\mu} \quad \text{and} \quad \mathscr{L}^{\ddagger} \mathscr{L} \equiv \mathscr{L} \mathscr{L}^{\ddagger} \equiv 1 e_0 \tag{2.1.10}$$

so that

$$(x'|y') = (\mathscr{L}^{*} x \mathscr{L} | \mathscr{L}^{*} y \mathscr{L}) = (x|y) \tag{2.1.11}$$

For the special case of space rotations we have $x^{0\prime} = x^0$ and $\mathscr{L}^{\ddagger} = \mathscr{L}^{*}$ (a quaternion). From equation (2.1.10) we see that (since in general $\mathscr{L}^{*} \neq \mathscr{L}^{\ddagger}$) the eight-part number $\mathscr{L}$ contains only six independent parameters as required. As examples we give explicitly the transformations R_{ϕ} and $\mathscr{L}_{v}$ for rotation through $+\phi$ about the $+z$ axis and a boost $+v$ along the $+x$ axis:

$$\begin{aligned} x' &= R_{\phi}^{*} x R_{\phi}, \qquad R_{\phi} = \cos(\phi/2)\, e_0 - i \sin(\phi/2)\, e_3 \\ x' &= \mathscr{L}_{v}^{*} x \mathscr{L}_{v}, \qquad \mathscr{L}_{v} = \cosh(\theta/2)\, e_0 + \sinh(\theta/2)\, e_1 \end{aligned} \tag{2.1.12}$$

where $\tanh\theta = (-v/c)$. We readily check that $R_{\phi}^{-1} = R_{\phi}^{\ddagger} = R_{-\phi}$ and $\mathscr{L}_{v}^{-1} = \mathscr{L}_{v}^{\ddagger} = \mathscr{L}_{-v}$.

Hypercomplex numbers, $a \equiv a^{\mu} e_{\mu}$, which transform as

$$a \to a' = \mathscr{L}^{\ddagger} a \mathscr{L} = a^0 e_0 + \mathscr{L}^{\ddagger} a^k e_k \mathscr{L}, \tag{2.1.13}$$

will appear often, and will be called axial 4-vectors. Scalars and space-like numbers are special cases.

The flat space-time differential operator can be defined as follows

$$\partial \equiv \partial^{\mu} e_{\mu}, \qquad \partial^{\mu} \equiv \partial/\partial x_{\mu} \qquad \text{and} \qquad \partial^{\ddagger} = \partial^{\mu} e_{\mu}^{\ddagger} = \partial_{\mu} e_{\mu} \tag{2.1.14}$$

It transforms as a 4-vector,

$$\partial \to \partial' = \mathscr{L}^{*} \partial \mathscr{L}, \qquad \text{and} \qquad (\partial'|\partial') = (\partial|\partial) \tag{2.1.15}$$

Our choice of $e_1 e_2 = +ie_3$ corresponds to a right-handed coordinate system. Note that $e_1^{\ddagger} e_2^{\ddagger} = -ie_3^{\ddagger}$ and therefore $x^{\ddagger}$ corresponds to a left-handed coordinate system. Similarly, we find

$$x^{\ddagger} \to x^{\ddagger\prime} = \mathscr{L}^{\ddagger} x^{\ddagger} \mathscr{L}^{\ddagger *} \tag{2.1.16}$$

The usual tensor treatment of vector analysis does not admit a direct generalization of axial vectors such as $\mathbf{r} \times \mathbf{p}$ when going to four-dimensional space-time. The totally antisymmetric angular momentum tensor has six non-zero components, and hence doesn't find a 'vector' representation. In the hypercomplex number formulation, however, we have

$$x \boxtimes y \equiv \tfrac{1}{2}(x^{\ddagger} y - y^{\ddagger} x) = -(x \boxtimes y)^{\ddagger} \tag{2.1.17}$$

where x and y are 4-vectors. Note that

$$x \boxtimes y \to x' \boxtimes y' = \mathscr{L}^{\ddagger}(x \boxtimes y)\mathscr{L} \tag{2.1.18}$$

so that $x \boxtimes y$ is an axial 4-vector. It is natural then to define an angular momentum axial 4-vector by

$$L \equiv x \boxtimes p \tag{2.1.19}$$

where x is the 'position' 4-vector and p is the momentum 4-vector. Similarly, we can define the hyper-curl as $\partial \boxtimes x$.

Some examples of how classical physics looks in this notation are given below:

Generalized momentum

$$\pi \equiv p + (e/c) A \tag{2.1.20}$$

Action

$$\pi \equiv -\partial S, \qquad S^{\ddagger} = S \tag{2.1.21}$$

Hamilton-Jacobi equation

$$\left(\partial S + \frac{e}{c} A \middle| \partial S + \frac{e}{c} A\right) = m^2 c^2 \tag{2.1.22}$$

Energy-momentum

$$(p|p) = m^2 c^2 \tag{2.1.23}$$

Electron equation

$$mc\ddot{x} = \frac{e}{c} \dot{x} \partial \boxtimes A + \frac{2e^2}{3c} [\dddot{x} + (\dot{x}|\dddot{x}) \dot{x}] \tag{2.1.24}$$

where $\dot{x} \equiv d/ds(x)$.

Electromagnetic fields

$$-E + iB \equiv \partial \boxtimes A, \qquad E \equiv E^k e_k, \quad B \equiv B^k e_k \tag{2.1.25}$$

Maxwell's equations

$$(\partial|\partial) A - \partial(\partial|A) = -J \tag{2.1.26}$$

2.2. *Curved Space-Time*

We consider now a non-flat space-time $[g_{\mu\nu} = g_{\mu\nu}(x)]$. To distinguish clearly between flat and curved space-time, we replace e_μ by $b_\mu(x)$ in writing 4-vectors such as

$$ds \equiv dx^\mu b_\mu \tag{2.2.1}$$

Then we have

$$(ds|ds) \equiv ds^2 \equiv \tfrac{1}{4}(b_\mu^{\ddagger} b_\nu + b_\mu b_\nu^{\ddagger} + b_\nu^{\ddagger} b_\mu + b_\nu b_\mu^{\ddagger}) dx^\mu dx^\nu$$

and

$$g_{\mu\nu}(x) \equiv \tfrac{1}{4}(b_\mu^{\ddagger} b_\nu + b_\mu b_\nu^{\ddagger} + b_\nu^{\ddagger} b_\mu + b_\nu b_\mu^{\ddagger}) \tag{2.2.2}$$

The metric tensor $g_{\mu\nu}$ must be Lorentz invariant and a scalar ($\propto e_0$), therefore we define

$$b_\mu{}^\ddagger b_\nu \equiv h^\lambda_{\mu\nu} e_\lambda \qquad \text{and} \qquad (b_\mu{}^\ddagger b_\nu)^\ddagger = b_\nu{}^\ddagger b_\mu \tag{2.2.3}$$

whereas $b_\mu{}^\ddagger$ is no longer simply related to b_μ, and $b_\mu b_\nu$ is no longer defined. We define

$$\begin{gathered} b_0 e_0 \equiv b_0, \quad b_1 e_1 \equiv b_0, \quad b_1 e_2 \equiv ib_3 \equiv -e_2 b_1, \\ b_1{}^\ddagger e_1 = -b_1{}^\ddagger e_1{}^\ddagger \equiv -b_0{}^\ddagger, \quad b_1{}^\ddagger e_2 = -b_1{}^\ddagger e_2{}^\ddagger \equiv +ib_3{}^\ddagger, \text{ etc.} \end{gathered} \tag{2.2.4}$$

We consider only transformations between reference frames of the form

$$ds \to ds' \equiv \mathscr{L}^* ds \mathscr{L} \tag{2.2.5}$$

where

$$\mathscr{L} \equiv \mathscr{L}^\mu e_\mu \qquad \text{and} \qquad \mathscr{L}^\ddagger \mathscr{L} \equiv \mathscr{L}\mathscr{L}^\ddagger \equiv 1 e_0 \tag{2.2.6}$$

Such transformations are well defined, since $e_\mu b_\nu e_\lambda$ is well defined. Such transformations are asymptotic to Lorentz transformations for events far from all matter and will be called free-fall transformations.

For a 4-vector $A(x)$ we define

$$A(x + dx) \equiv A(x) + dA \tag{2.2.7}$$

where

$$d(A(x)) = d(A^\mu b_\mu) \equiv (dA^\mu) b_\mu + A^\mu db_\mu \equiv (DA^\mu) b_\mu \tag{2.2.8}$$

To first order we can define

$$db_\mu \equiv \Gamma^\nu_{\mu\lambda} dx^\lambda b_\nu \equiv g_{\eta\lambda} \Gamma^{\nu\eta}_\mu dx^\lambda b_\nu \equiv (\Gamma_\mu{}^\nu | ds) b_\nu \tag{2.2.9}$$

where $\Gamma_\mu{}^\nu \equiv \Gamma^{\nu\lambda}_\mu b_\lambda$ and $ds \equiv dx^\mu b_\mu$. Here, the Christoffel-like symbols $\Gamma^{\nu\lambda}_\mu$ will be defined to be contravariant components of the 4-vector $\Gamma_\mu{}^\nu$, so that db_μ is invariant (as is b_μ).

We have then

$$d(A(x)) = D(A^\nu) b_\nu = [dA^\nu + A^\mu (\Gamma_\mu{}^\nu | ds)] b_\nu \tag{2.2.10}$$

The covariant differential operator $\mathscr{D}$ is defined by

$$\mathscr{D} \equiv \mathscr{D}^\eta b_\eta \qquad \text{and} \qquad \mathscr{D}^\eta(A^\nu) \equiv \frac{\partial A^\nu}{\partial x_\eta} + A^\mu \Gamma^{\nu\eta}_\mu \tag{2.2.11}$$

We further define $\mathscr{D}^\eta(\mathscr{L}^\nu) \equiv 0$, $\mathscr{D}^\eta((\Gamma_\mu{}^\nu | ds)) \equiv 0$, $\mathscr{D}^\eta(g_{\mu\nu}) \equiv 0$, $\mathscr{D}^\eta(b_\mu) \equiv 0$, $\mathscr{D}^\eta(b_\mu{}^\ddagger) = 0$, and $(db_\mu)^\ddagger \equiv d(b_\mu{}^\ddagger)$. We should also have $\mathscr{D}' = \mathscr{L}^* \mathscr{D} \mathscr{L}$.

We can see now why $b_\mu b_\nu$ is not defined. Otherwise, consistency difficulties would arise with things like $d(b_\mu b_\nu)$.

For warped space-time, $A^\ddagger B$ and $AB^\ddagger$ are not equivalent, and we define a second invariant product

$$[A|B] \equiv \tfrac{1}{4}[A^\ddagger B - AB^\ddagger + B^\ddagger A - BA^\ddagger] \tag{2.2.12}$$

This gives rise to the warp operator

$$[\mathscr{D}|\mathscr{D}] \equiv \tfrac{1}{2}[\mathscr{D}^\ddagger \mathscr{D} - \mathscr{D}\mathscr{D}^\ddagger] \tag{2.2.13}$$

In the flat space-time limit $[A|B]$ goes to zero.

Thus, $[\mathscr{D}|\mathscr{D}]$ is a likely choice for a warp-inducing gravitational source equation.

3. *Hypercomplex Constants*

The link between algebraic number theory and reality is very profound, as evidenced by our ability to describe and predict physical phenomena using equations. If we take the point of view that the ring of complex quaternions gives the natural number system of space-time, we are naturally led to ask: 'What is the natural number system which characterizes matter and energy in this space-time?' We have seen that the classical laws and their covariance can be simply expressed using complex quaternions. What about the fundamental constants themselves? If a fundamental 'constant' K transforms as an axial 4-vector,

$$K = k^{\mu} e_{\mu} \rightarrow K' = k^{\mu\prime} e_{\mu} = \mathscr{L}^{\ddagger} K \mathscr{L} = k^0 e_0 + \mathscr{L}^{\ddagger} k^j e_j \mathscr{L} \tag{3.1}$$

then $K^{\ddagger} K = \{(k_0)^2 - [(k^1)^2 + (k^2)^2 + (k^3)^2]\} e_0$ is invariant and AK transforms as a 4-vector if A does. Thus covariant field equations are easily constructed in this formalism with axial 4-vector coupling parameters such as charge e, mass m, Planck's constant $\hbar$, and Newton's constant G.

4. *Basic Wave Equations*

In formulating covariant field equations using the hypercomplex number formalism developed in Section 2, three types of fields arise quite naturally. Their transformation properties are summarized below

$$\psi_s \rightarrow \psi_s' = \psi_s, \qquad \psi_s = \psi_s{}^0 e_0 \tag{4.1}$$

$$\psi_v \rightarrow \psi_v' = \mathscr{L}^* \psi_v, \qquad \psi_v = \psi_v{}^{\mu} e_{\mu} \tag{4.2}$$

$$\psi_a \rightarrow \psi_a' = \mathscr{L}^{\ddagger} \psi_a, \qquad \psi_a = \psi_a{}^{\mu} e_{\mu} \tag{4.3}$$

The basic postulate for generating quantum equations is that the classical connonical momentum $p \rightarrow i\hbar\partial$.

If $\hbar' = \mathscr{L}^{\ddagger} \hbar \mathscr{L}$, we can modify this to $p \rightarrow i\partial\hbar$, where $\partial^{\mu} \hbar^{\nu} \equiv \hbar^{\nu} \partial^{\mu}$. Then

$$i\partial' \hbar' = i\mathscr{L}^* \partial \mathscr{L} \mathscr{L}^{\ddagger} \hbar \mathscr{L} = \mathscr{L}^* \, i\partial\hbar \mathscr{L} \tag{4.4}$$

and

$$i\hbar^{\ddagger\prime} \partial^{\ddagger\prime} = i\mathscr{L}^{\ddagger} \hbar^{\ddagger} \mathscr{L} \mathscr{L}^{\ddagger} \partial^{\ddagger} \mathscr{L}^{\ddagger *} = \mathscr{L}^{\ddagger} \, i\hbar^{\ddagger} \partial^{\ddagger} \mathscr{L}^{\ddagger *} \tag{4.5}$$

The simplest zero-mass equations are

$$i\,\partial\hbar\psi_s = 0 \tag{4.6}$$

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_s = 0 \tag{4.7}$$

$$i\,\partial\hbar\psi_a = 0 \tag{4.8}$$

$$i\hbar^{\ddagger}\,\partial^{\ddagger}\,\psi_v = 0 \tag{4.9}$$

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_v = 0 \tag{4.10}$$

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_a = 0 \tag{4.11}$$

and

$$(i\,\partial\hbar|i\,\partial\hbar)\,A - i\,\partial\hbar(i\,\partial\hbar|A) = 0 \tag{4.12}$$

If $m' = \mathscr{L}^{\ddagger} m \mathscr{L}$, we can form the following equations:

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_s = (mc|mc)\,\psi_s \tag{4.13}$$

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_v = (mc|mc)\,\psi_v \tag{4.14}$$

$$(i\,\partial\hbar|i\,\partial\hbar)\,\psi_a = (mc|mc)\,\psi_a \tag{4.15}$$

and the coupled equations

$$i\,\partial\hbar\psi_a = m^*\,c\psi_v \tag{4.16a}$$

$$i\hbar^{\ddagger}\,\partial^{\ddagger}\,\psi_v = mc\psi_a \tag{4.16b}$$

Equations (4.16a) and (4.16b) are very similar to the tensor formulation of the Dirac equation. Here, however, $\psi_a = \psi_a{}^{\mu} e_{\mu}$ and $\psi_v = \psi_v{}^{\mu} e_{\mu}$, which gives eight complex functions instead of four.

In flat space-time considered here, equations (4.16a) and (4.16b) can be converted to independent 'Klein-Gordon' equations, provided $m = m^{\ddagger}$ and $\hbar = \hbar^{\ddagger}$

$$i\hbar^{\ddagger}\,\partial^{\ddagger}\,i\,\partial\hbar\psi_a = m^*\,mc^2\,\psi_a \tag{4.17}$$

and

$$i\,\partial\hbar i\hbar^{\ddagger}\,\partial^{\ddagger}\,\psi_v = mm^*\,c^2\,\psi_v \tag{4.18}$$

These are essentially equivalent to equation (4.15) and equation (4.14) respectively, if $m = m^*$.

Since equations (4.16a) and (4.16b) differ from the Dirac equation, it is of some interest to examine their classical and non-relativistic limits for comparison. If in the classical limit we assume that $\psi_a \to \exp(iS/\hbar)$, $\psi_v{}^{\ddagger}\psi_v \to \exp(2iS/\hbar)$, $\hbar^{\ddagger} \to \hbar$, and $m^{\ddagger} \to m$; then multiplication of equation (4.16a) by its hyperconjugate from the right gives

$$(\partial S)(\partial S)^{\ddagger} = m^2\,c^2 \tag{4.19}$$

If equations (4.16a) and (4.16b) are modified by the usual coupling to the electromagnetic field, $i\partial\hbar \to \pi \equiv i\partial\hbar - A(e/c)$, then the same procedure (assuming $e^{\ddagger} \to e$) yields essentially the Hamilton-Jacobi equation, equation (2.1.22)

$$(\partial S + (e/c)A)(\partial S + (e/c)\,A)^{\ddagger} = m^2\,c^2 \tag{4.20}$$

The non-relativistic limit is obtained from equation (4.17). We replace $i\partial\hbar$ by π, the generalized momentum, for comparison with the Pauli equation and assume $h^{\ddagger} \to h, m^{\ddagger} \to m, e^{\ddagger} \to e$. To obtain the non-relativistic limit we take out the rapid time variation of ψ_a,

$$\psi_a \equiv \exp\left(-i\frac{mc^2}{\hbar}t\right)\tilde{\psi}_a \tag{4.21}$$

and then assume the terms containing mc dominate all others. The result is

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}_a \approx \left[\sum_k \frac{\pi^k\pi^k}{2m} + i\frac{e\hbar}{2mc}(\partial^{\ddagger} \boxtimes A) + eA^0\right]\tilde{\psi}_a \tag{4.22}$$

which can also be written

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}_a \approx \left[\sum_k \frac{(i\hbar\,\partial^k - (e/c)\,A^k)^2}{2m} - \frac{e\hbar}{2mc}B + e\phi + i\frac{e\hbar}{2mc}E\right]\tilde{\psi}_a \tag{4.23}$$

Here, $A^0 \equiv \phi$, $B \equiv B^k e_k$ and $E \equiv E^k e_k$. The term involving E, it has been argued (Condon & Shortley, 1935), is of order $(v^2/c^2)\phi$, and therefore can be neglected. Since $e_k \leftrightarrow \sigma_k$, equation (4.23) would be isomorphic to the Pauli equation, except that $\tilde{\psi}_a = \tilde{\psi}_a{}^{\mu} e_{\mu}$, and has four, instead of two, independent parts.

We now define the spin operators

$$S_k \equiv \tfrac{1}{2}\hbar e_k \tag{4.24}$$

and the raising and lowering operators

$$S_+ \equiv \tfrac{1}{2}(S_x + iS_y), \qquad S_- \equiv \tfrac{1}{2}(S_x - iS_y) \tag{4.25}$$

We find that both $(e_1 + ie_2)$ and $(e_0 + e_3)$ are spin-up eigenstates of S_z, both $(e_1 - ie_2)$ and $(e_0 - e_3)$ are spin-down states, and that the raising and lowering operators connect the states as follows

$$(e_1 + ie_2) \leftrightarrow (e_0 - e_3), \qquad (e_0 + e_3) \leftrightarrow (e_1 - ie_2)$$

We therefore obtain the Pauli equation by the two-to-one identification, $(e_1 + ie_2)$ and $(e_0 + e_3)$ go to spin-up. Similarly, for spin-down. Thus for $m^{\ddagger} = m$ and $h^{\ddagger} = \hbar$, the extra degrees of freedom in ψ are probably redundant.

We note in concluding this discussion of spin states that

$$(e_0 + e_3)^{\ddagger *} = +(e_0 - e_3) \tag{4.26}$$

whereas

$$(e_1 + ie_2)^{\ddagger *} = -(e_1 - ie_2) \tag{4.27}$$

We look now briefly at the free particle solutions of equations (4.16a) and (4.16b) for $m^{\ddagger} = m^* = m$ and $h^{\ddagger} = \hbar^* = \hbar$. We define the energy operator $i\partial^0\hbar$ and the kth component momentum operator $i\partial^k\hbar$, $k = 1, 2, 3$. We further combine ψ_a and ψ_v into a single matrix,

$$\psi \equiv \begin{pmatrix}\psi_a \\ \psi_v\end{pmatrix} \tag{4.28}$$

The rest state solution is by definition given by

$$i\partial^k \hbar\psi_{\text{rest}} \equiv 0, \qquad k = 1,2,3 \tag{4.29}$$

We then can obtain uniform velocity states in the usual way by Lorentz transformation

$$\psi(x) \equiv \begin{pmatrix}\psi_a(x)\\ \psi_v(x)\end{pmatrix} \rightarrow \psi'(x') = \begin{pmatrix}\psi_a'(x')\\ \psi_v'(x')\end{pmatrix} = \begin{pmatrix}\mathscr{L}^\ddagger \psi_a(\mathscr{L}^{\ddagger *} x' \mathscr{L}^\ddagger)\\ \mathscr{L}^* \psi_v(\mathscr{L}^{\ddagger *} x' \mathscr{L}^\ddagger)\end{pmatrix} \tag{4.30}$$

where $x \rightarrow x' = \mathscr{L}^* x \mathscr{L}$. We define the parameters $p^{\mu\prime}$ by

$$p' \equiv p^{\mu\prime} e_\mu = \mathscr{L}^* mc \mathscr{L} \tag{4.31}$$

and let $p^{0\prime} \equiv E'/c$. This gives the usual formulas,

$$E' = \gamma mc^2 \qquad \text{and} \qquad p^{1\prime} = \gamma mv \tag{4.32}$$

for transformation to a frame moving along $-x$ with speed $-v$.

The eight independent rest solutions of equations (4.16a) and (4.16b) for arbitrary spin can be written as

$$\psi_{+mc^2} = \exp\left[-\frac{i}{\hbar}(p^0 x^0)\right]\begin{pmatrix}a^\mu e_\mu\\ a^\mu e_\mu\end{pmatrix} \tag{4.33}$$

and

$$\psi_{-mc^2} = \exp\left[+\frac{i}{\hbar}(p^0 x^0)\right]\begin{pmatrix}a^\mu e_\mu\\ -a^\mu e_\mu\end{pmatrix} \tag{4.34}$$

where $p^0 = mc$, $x^0 = ct$ and a^μ is arbitrary. Operation on these functions with the energy operator $i\partial^0 \hbar$ shows that they are states of positive and negative energy as indicated by the subscripts. The corresponding plane wave states are given by

$$\psi_{\pm mc^2} \rightarrow \psi_{\pm E} = \exp\left[\mp\frac{i}{\hbar}(p|x)\right]\begin{pmatrix}\mathscr{L}^\ddagger a\\ \pm\mathscr{L}^* a\end{pmatrix} \tag{4.35}$$

where $a \equiv a^\mu e_\mu \equiv b_+{}^1(e_1 + ie_2) + b_+{}^2(e_0 + e_3) + b_-{}^1(e_0 - e_3) + b_-{}^2(e_1 - ie_2)$.

The PCT transformation properties of equations (4.16a) and (4.16b) can also be examined in the usual way (Brown, 1962).

5. *Wavefunction Transformation*

If instead of equations (4.2) and (4.3) we try

$$\psi_a \rightarrow \psi_a' = \mathscr{L}^\ddagger \psi_a \mathscr{L}, \qquad \psi_v \rightarrow \psi_v' = \mathscr{L}^* \psi_v \mathscr{L}, \qquad m \rightarrow m' = \mathscr{L}^\ddagger m \mathscr{L}$$
$$\text{and} \qquad \hbar \rightarrow \hbar' = \mathscr{L}^\ddagger \hbar \mathscr{L} \tag{5.1}$$

then the following equation is covariant

$$\begin{aligned} i\partial\hbar\psi_a &= \psi_v mc \\ i\hbar^\ddagger \partial^\ddagger \psi_v &= \psi_a m^\ddagger c \end{aligned} \tag{5.2}$$

Here $\partial \equiv e_\mu \partial^\mu$, $\hbar \equiv e_\mu \hbar^\mu$, $m \equiv e_\mu m^\mu$, $\psi_a = e_\mu \psi_a{}^\mu$, and $\psi_v = e_\mu \psi_v{}^\mu$, as before. It also follows directly that

$$\bar{\psi}\psi \equiv \psi_a{}^* \psi_v + \psi_v{}^* \psi_a \equiv (\bar{\psi}\psi)^\mu e_\mu \tag{5.3}$$

transforms like a 4-vector. However, direct calculation (taking $\hbar = \hbar^\ddagger = \hbar^*$, $m = m^\ddagger = m^*$, and making use of the wave equation) shows that for

$$(\partial/\bar{\psi}\psi) \equiv \tfrac{1}{2}(\partial^\ddagger(\bar{\psi}\psi) + [\partial^\ddagger(\bar{\psi}\psi)]^\ddagger) = \partial^0[(\bar{\psi}\psi)^0] - \partial^k[(\bar{\psi}\psi)^k] \tag{5.4}$$

we have

$$\partial^k(\bar{\psi}\psi)^k = 0 \tag{5.5}$$

Thus $\bar{\psi}\psi$ does not meet the requirements of a conserved probability current. We then go back to the other possibility, which is

$$\psi_a \to \psi_a{}' = \mathscr{L}^\ddagger \psi_a \qquad \text{and} \qquad \psi_v \to \psi_v{}' = \mathscr{L}^* \psi_v \tag{5.6}$$

The corresponding wave equation is

$$\begin{aligned} i\partial\hbar\psi_a &= m^* c\psi_v \\ i\hbar^\ddagger \partial^\ddagger \psi_v &= mc\psi_a \end{aligned} \tag{5.7}$$

which is covariant since

$$m^* \to m^{*\prime} = (\mathscr{L}^\ddagger m \mathscr{L})^* = \mathscr{L}^* m^* \mathscr{L}^{\ddagger *} \tag{5.8}$$

We then find that in addition to the invariant $\psi_a{}^\ddagger \psi_a + \psi_v{}^\ddagger \psi_v$, we have the invariant $\psi_a{}^* \psi_v + \psi_v{}^* \psi_a$. In matrix notation we have

$$\begin{pmatrix} 0 & i\hbar^\ddagger \partial^\ddagger \\ i\partial h & 0 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} = \begin{pmatrix} mc & 0 \\ 0 & m^* c \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \tag{5.9}$$

Let

$$\psi \equiv \begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \qquad \text{and} \qquad c\hat{m} \equiv \begin{pmatrix} 0 & i\hbar^\ddagger \partial^\ddagger \\ i\hbar\partial & 0 \end{pmatrix} \tag{5.10}$$

then

$$c\hat{m}\psi = \begin{pmatrix} mc & 0 \\ 0 & m^* c \end{pmatrix} \psi \to c\hat{m}' \psi' = \begin{pmatrix} m' c & 0 \\ 0 & m'^* c \end{pmatrix} \psi' \tag{5.12}$$

and

$$\begin{pmatrix} m' & 0 \\ 0 & m'^* \end{pmatrix} = \begin{pmatrix} \mathscr{L}^\ddagger & 0 \\ 0 & \mathscr{L}^* \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m^* \end{pmatrix} \begin{pmatrix} \mathscr{L} & 0 \\ 0 & \mathscr{L}^{*\ddagger} \end{pmatrix} \tag{5.13}$$

We also have

$$\bar{\psi}\psi = \psi^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi \right]^* \psi \tag{5.14}$$

which is similar to $\bar{\psi}\psi$ in the usual Dirac theory. We note that

$$(\bar{\psi}\psi)^* = (\bar{\psi}\psi) \neq (\bar{\psi}\psi)^\ddagger \qquad \text{and} \qquad (\psi^\ddagger \psi) = (\psi^\ddagger \psi)^\ddagger \neq (\psi^\ddagger \psi)^* \tag{5.15}$$

From equation (5.7) we see also that ψ_a does not satisfy a Klein-Gordon equation unless $m = m^\ddagger$, since $\partial^\ddagger$ and m^* would not commute.

6. *Conserved Probability Current*

Starting with equation (5.7) and considering $\hbar$ and m as hyper-numbers we obtain by suitable manipulation

$$\partial^0(\psi_a{}^*)[e_0\hbar]^*\psi_a+\psi_a{}^*[e_0\hbar]\,\partial^0(\psi_a)+\partial^0(\psi_v{}^*)[\hbar^\ddagger e_0{}^\ddagger]^*\psi_v+\psi_v{}^*[\hbar^\ddagger e_0{}^\ddagger]\,\partial^0(\psi_v)$$
$$=-\partial^k(\psi_a{}^*)[e_k\hbar]^*\psi_a-\psi_a{}^*[e_k\hbar]\,\partial^k(\psi_a)-\partial^k(\psi_v{}^*)[\hbar^\ddagger e_k{}^\ddagger]^*\psi_v$$
$$-\psi_v{}^*[\hbar^\ddagger e_k{}^\ddagger]\,\partial^k(\psi_v) \quad (6.1)$$

If we now assume that $\hbar=\hbar^*=\hbar^\ddagger$, then we have

$$\hbar\,\partial^0(\psi^*\psi)+\hbar\,\partial^k\left[\psi^*\begin{pmatrix}e_k & 0\\ 0 & e_k{}^\ddagger\end{pmatrix}\psi\right]=0 \quad (6.2)$$

(No restriction on m was needed to obtain this result!)

We therefore define the hyper-number function $\tilde{j}$ as

$$\tilde{j}\equiv\tilde{j}^\mu e_\mu,\qquad \tilde{j}^\mu\equiv c\psi^*\begin{pmatrix}e_\mu{}^\ddagger & 0\\ 0 & e_\mu\end{pmatrix}\psi,\qquad \tilde{\rho}\equiv\tilde{j}^0/c \quad (6.3)$$

and obtain

$$\partial^0\tilde{j}^0-\partial^k\tilde{j}^k=0\qquad\text{or}\qquad \frac{\partial\tilde{\rho}}{\partial t}=-\frac{\partial}{\partial x^k}\tilde{j}^k \quad (6.4)$$

Because of the covariance of the wave equation, we also have

$$\partial^{0\prime}\tilde{j}^{0\prime}-\partial^{k\prime}\tilde{j}^{k\prime}=0 \quad (6.5)$$

Therefore $\tilde{j}$ should transform as a 4-vector. Explicitly

$$\tilde{j}'=\psi^{*\prime}\begin{pmatrix}e_\mu{}^\ddagger & 0\\ 0 & e_\mu\end{pmatrix}\psi' e_\mu=\psi^*\begin{pmatrix}\mathscr{L}^{\ddagger *} & 0\\ 0 & \mathscr{L}\end{pmatrix}\begin{pmatrix}e_\mu{}^\ddagger & 0\\ 0 & e_\mu\end{pmatrix}\begin{pmatrix}\mathscr{L}^\ddagger & 0\\ 0 & \mathscr{L}^*\end{pmatrix}\psi e_\mu \quad (6.6)$$

But $\tilde{j}$ is not a satisfactory probability current, because $\psi^*\psi\neq(\psi^*\psi)^\ddagger$. We therefore define

$$j\equiv\frac{c}{2}\left[\psi^*\begin{pmatrix}e_\mu{}^\ddagger & 0\\ 0 & e_\mu\end{pmatrix}\psi+\left(\psi^*\begin{pmatrix}e_\mu{}^\ddagger & 0\\ 0 & e_\mu\end{pmatrix}\psi\right)^\ddagger\right]e_\mu \quad (6.7)$$

and

$$\rho\equiv\tfrac{1}{2}[\psi^*\psi+(\psi^*\psi)^\ddagger]=|\psi_a{}^0|^2+|\psi_a{}^1|^2+|\psi_a{}^2|^2+|\psi_a{}^3|^2$$
$$+|\psi_v{}^0|^2+|\psi_v{}^1|^2+|\psi_v{}^2|^2+|\psi_v{}^3|^2 \quad (6.8)$$

From equations (6.4) and (6.7) we verify that

$$\partial^0 j^0-\partial^k j^k=0 \quad (6.9)$$

A very tedious explicit calculation confirms that

$$j'=\mathscr{L}^* j\mathscr{L}=j^{\mu\prime}e_\mu \quad (6.10)$$

Here $j^{0\prime}$ consists of 128 terms and $j^{k\prime}$ consists of 32 terms. That j is not easily shown to be a 4-vector is somewhat disturbing. However, the covariance of equation (6.9) is very convincing evidence in itself. Hereafter we shall assume that $\hbar=\hbar^\ddagger=\hbar^*$ so that j can be interpreted as the probability current (ρ is real and positive). Therefore $\hbar'=\mathscr{L}^\ddagger\hbar\mathscr{L}=\hbar\mathscr{L}^\ddagger\mathscr{L}=\hbar$ but $m'=\mathscr{L}^\ddagger m\mathscr{L}\neq m$ in general.

7. *Inner Product*

We now define the innerproduct as

$$\langle\psi|\phi\rangle \equiv \int \tfrac{1}{2}\,[\psi^*\phi + (\psi^*\phi)^\ddagger]\,d^3x \tag{7.1}$$

Therefore

$$\rho = \tfrac{1}{2}[\psi^*\psi + (\psi^*\psi)^\ddagger] = \rho^* = \rho^\ddagger \tag{7.2}$$

The following properties are easily verified:

$$\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle \tag{7.3}$$

$$\langle a\psi|\phi\rangle = a^*\langle\psi|\phi\rangle = \langle\psi|a^*\phi\rangle \quad \text{for } a = a^\ddagger \tag{7.4}$$

$$\langle\psi|a\phi\rangle = a\langle\psi|\phi\rangle = \langle a^*\psi|\phi\rangle \quad \text{for } a = a^\ddagger \tag{7.5}$$

$$\langle\psi|\phi\rangle^\ddagger = \langle\psi|\phi\rangle = \langle\phi^{\ddagger *}|\psi^{\ddagger *}\rangle \tag{7.6}$$

and

$$\langle\psi|a\phi\rangle = \langle a^*\psi|\phi\rangle \quad \text{for } a \neq a^\ddagger \text{ or } a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{7.7}$$

We can define the adjoint as follows

$$\langle\psi|\underset{\rightarrow}{\hat{A}}\phi\rangle \equiv \langle\underset{\rightarrow}{\hat{A}}{}^\dagger\psi|\phi\rangle \tag{7.8}$$

Thus for the mass operator defined in equation (5.10) we have

$$\langle\mathscr{m}\psi|\psi\rangle = \left\langle\begin{pmatrix} m & 0 \\ 0 & m^* \end{pmatrix}\psi\middle|\psi\right\rangle = \left\langle\psi\middle|\begin{pmatrix} m^* & 0 \\ 0 & m \end{pmatrix}\psi\right\rangle \equiv \left\langle\psi\middle|\begin{pmatrix} m^* & 0 \\ 0 & m \end{pmatrix}\middle|\psi\right\rangle \tag{7.9}$$

and

$$\langle\psi|\mathscr{m}\psi\rangle = \left\langle\psi\middle|\begin{pmatrix} m & 0 \\ 0 & m^* \end{pmatrix}\psi\right\rangle = \left\langle\psi\begin{pmatrix} m & 0 \\ 0 & m^* \end{pmatrix}\middle|\psi\right\rangle \tag{7.10}$$

Therefore, $\mathscr{m} \neq \mathscr{m}^\dagger$ unless $m = m^*$. But if $m' = \mathscr{L}^\ddagger m \mathscr{L}$, then in general

$$m'^* = \mathscr{L}^* m^* \mathscr{L}^{\ddagger *} \tag{7.11}$$

so that $m \equiv m^* \Rightarrow m' = m'^*$ only for space rotations. Recall that $R^\ddagger = R^*$ (a quaternion) for such rotations but in general $\mathscr{L}^\ddagger \neq \mathscr{L}^*$. For the special case $m = m^\ddagger$ (scalar mass) we have $m \equiv m^* \Rightarrow m' = m'^* = m$ and then $\mathscr{m}^\dagger = \mathscr{m}$. When $m \neq m^\ddagger$ the wave equation becomes a kind of generalized eigen-problem with both ψ and $m(\psi)$ to be determined:

$$\mathscr{m}\psi_m = \begin{pmatrix} m(\psi) & 0 \\ 0 & [m(\psi)]^* \end{pmatrix}\psi_m \tag{7.12}$$

The other extreme case for the mass is $m \equiv -m^\ddagger$. Then

$$m'^\ddagger = (\mathscr{L}^\ddagger m \mathscr{L})^\ddagger = \mathscr{L}^\ddagger m^\ddagger \mathscr{L} = -\mathscr{L}^\ddagger m \mathscr{L} = -m' \tag{7.13}$$

and

$$m'^\ddagger m' = (\mathscr{L}^\ddagger m \mathscr{L})^\ddagger(\mathscr{L}^\ddagger m \mathscr{L}) = m^\ddagger m \mathscr{L}^\ddagger \mathscr{L} = m^\ddagger m \tag{7.14}$$

Hence

$$m^{\ddagger} m = -[(m')^2 + (m^2)^2 + (m^3)^2] \tag{7.15}$$

gives a scalar representing the mass of such particles. It is very interesting to note that for $m = -m^{\ddagger}$ the two parts of the wave equation become degenerate and

$$\psi_v = \psi_a^{\ddagger *} (\text{quaternion conjugates}) \tag{7.16}$$

We have then

$$i\hbar\, \partial \psi_a = m^* \psi_a^{\ddagger *} \tag{7.17}$$

when $m = -m^{\ddagger}$, and for a general state m is complex. If $m^{\ddagger} m$ were very small but finite, this wave equation could apply perhaps to neutrinos and if so, account easily for there being more than one kind by assigning different values of $m^{\ddagger} m$ to them.

For the mass expectation value we have

$$\langle \psi | \mathscr{m} | \psi \rangle = \int \tfrac{1}{2} [\psi^* \mathscr{m} \psi + (\psi^* \mathscr{m} \psi)^{\ddagger}]\, d^3 x \tag{7.18}$$

When $m = -m^{\ddagger}$, so that $\psi_v = \psi_a^{\ddagger *}$, this reduces to

$$\begin{aligned} \langle \psi | \mathscr{m} | \psi \rangle &= \int \tfrac{1}{2} [\psi_a{}^*(m + m^{\ddagger *})\, \psi_a + \psi_a{}^{\ddagger}(m^* + m^{\ddagger})\, \psi_a^{\ddagger *}]\, d^3 x \\ &= \int \tfrac{1}{2} [\psi_a{}^*(m - m^*)\, \psi_a - \psi_a{}^{\ddagger}(m - m^*)\, \psi_a^{\ddagger *}]\, d^3 x \\ &= - \langle \psi | \mathscr{m} | \psi \rangle^* \end{aligned} \tag{7.19}$$

Thus if ψ_a is a quaternion state ($\psi_a = \psi_a^{\ddagger *}$) the mass expectation value is zero, otherwise it is imaginary. But ψ_a' would also be a quaternion only for rotation transformations, therefore the mass expectation value would in general be a non-zero imaginary number for particles with $m = -m^{\ddagger}$. The invariant $m^{\ddagger} m$ could be chosen real and would characterize the particle in an invariant way.

The assumption that $m \neq m^{\ddagger}$, has as its most drastic effect the negation of the superposition principle. This of itself is not sufficient justification for its rejection, however. Agreement with experiment must be the final test.

8. *Plan Waves and Hyper-Mass*

We show now how to find the analogue of plane wave states for a free particle with $m = m^{\mu} e_{\mu}$. We define energy and momentum operators as

$$\mathscr{E} \equiv i\hbar c\, \partial^0, \qquad \mathscr{P}^k \equiv i\hbar\, \partial^k = -i\hbar \frac{\partial}{\partial x^k} \tag{8.1}$$

and define the rest state ψ_m by $m = \mathfrak{m}$ and

$$\mathscr{P}^k \psi_{\mathfrak{m}} = 0, \qquad k = 1, 2, \text{or } 3 \tag{8.2}$$

Thus we have

$$\mathscr{m}\psi_{\mathrm{m}} = \begin{pmatrix} \mathrm{m} & 0 \\ 0 & \mathrm{m}^* \end{pmatrix} \psi_{\mathrm{m}} \tag{8.3}$$

or equivalently

$$i\hbar\,\partial^0 \psi_{am} = \mathrm{m}^*\, c\, \psi_{vm} \qquad \text{and} \qquad i\hbar\,\partial^0 \psi_{vm} = \mathrm{m} c \psi_{am} \tag{8.4}$$

We try a solution of the form

$$\psi_{am} \equiv \exp\left(-\frac{i}{\hbar}\xi c^2 t\right) a^\mu e_\mu \tag{8.5}$$

where

$$\xi \equiv \xi^\mu e_\mu = \xi^0 e_0 + \boldsymbol{\xi} \qquad \text{and} \qquad \xi^{\mu *} = \xi^\mu \tag{8.6}$$

Direct power series expansion confirms that

$$\exp(\pm i\theta\boldsymbol{\xi}) = \cos(|\boldsymbol{\xi}|\theta) \pm i\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\sin(|\boldsymbol{\xi}|\theta) \tag{8.7}$$

where

$$|\boldsymbol{\xi}| \equiv +\sqrt{(\boldsymbol{\xi}\boldsymbol{\xi})} = \sqrt{(\xi^k \xi^k)} \tag{8.8}$$

Substitution of equation (8.5) into equation (8.4) shows then that

$$\psi_{vm} = \eta\psi_{am}, \qquad \xi\xi = \mathrm{m}^*\mathrm{m} \qquad \text{and} \qquad \eta = \frac{\mathrm{m}^\ddagger \xi}{\mathrm{m}^\ddagger \mathrm{m}} \tag{8.9}$$

Along with equation (8.5) this gives a solution for equation (8.3). If the rest mass $m^* = m$, we have

$$\xi = \pm\mathrm{m} = \pm(\mathrm{m}^0 + \boldsymbol{\mathrm{m}}) \qquad \text{and} \qquad \eta = \pm 1 \tag{8.10}$$

Transformation to a reference frame moving with velocity $-v$ along $-x^1$ shows that a plane wave with velocity parameter $+v$ along $+x^1$ is given by

$$\psi_\pm = \begin{pmatrix} L^\ddagger \exp\left[-\frac{i}{\hbar}(\gamma\xi c^2 t - \gamma\xi v x^1)\right] a^\mu e_\mu \\ \pm L^* \exp\left[-\frac{i}{\hbar}(\gamma\xi c^2 t - \gamma\xi v x^1)\right] a^\mu e_\mu \end{pmatrix} \tag{8.11}$$

where

$$L \equiv (\tfrac{1}{2}(\gamma+1))^{1/2} e_0 + (\tfrac{1}{2}(\gamma-1))^{1/2} e_1 \qquad \text{and} \qquad \gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{8.12}$$

Direct substitution verifies that $\psi_\pm$ is a solution of

$$\mathscr{m}\psi = \begin{pmatrix} L^\ddagger \mathrm{m} L & 0 \\ 0 & (L^\ddagger \mathrm{m} L)^* \end{pmatrix} \psi \tag{8.13}$$

and that

$$m \equiv L^\ddagger \mathrm{m} L = \mathrm{m}^0 + \mathrm{m}^1 e_1 + \gamma\left(\mathrm{m}^2 + i\frac{v}{c}\mathrm{m}^3\right) e_2 + \gamma\left(\mathrm{m}^3 - i\frac{v}{c}\mathrm{m}^2\right) e_3 \tag{8.14}$$

By suitably orienting our coordinates in the rest frame we can make $m^2 = m^3 = 0$. Then for this plane wave, $\psi_\pm$, we have $m = \mathfrak{m}$. Note also that for this special case $[L, e_1] = 0$ and $[\xi, e_1] = 0$, so that $\psi_\pm$ can be an eigenstate of the spin operator $\frac{1}{2}\hbar e_1$. Direct calculation, using

$$L^\ddagger L = 1 \qquad \text{and} \qquad \frac{d}{d\theta}\exp(if(\theta)\,\boldsymbol{\xi}) = \exp(if(\theta)\,\boldsymbol{\xi})\,i\frac{df}{d\theta} \tag{8.15}$$

confirms that $\psi_\pm$ is a generalized eigenstate of the energy, $\hat{\mathscr{E}}$, and momentum, $\hat{\mathscr{P}}^k$, operators as well as the mass operator, $\hat{\mathfrak{m}}$

$$\hat{\mathscr{E}}\psi_\pm = \begin{pmatrix} \gamma L^\ddagger \xi L c^2 & 0 \\ 0 & (\gamma L^\ddagger \xi L c^2)^* \end{pmatrix}\psi_\pm \qquad \hat{\mathscr{P}}^k\psi_\pm = \begin{pmatrix} \gamma L^\ddagger \xi L v & 0 \\ 0 & (\gamma L^\ddagger \xi L v)^* \end{pmatrix}\psi_\pm$$

$$\hat{\mathscr{P}}^2\psi_\pm = 0 \qquad \text{and} \qquad \hat{\mathscr{P}}^3\psi_\pm = 0 \tag{8.16}$$

This gives the eigenvalues

$$E = \gamma L^\ddagger \xi L c^2 = \pm\gamma m c^2 \qquad \text{and} \qquad p^1 = \gamma L^\ddagger \xi L v = \pm\gamma m v \tag{8.17}$$

where m is given by equation (8.14).

The rest state norm gives, using equation (6.8),

$$\begin{aligned}\langle\psi_{\mathfrak{m}}|\psi_{\mathfrak{m}}\rangle &= \int \rho_{\text{rest}}\, d^3x = 2[|a^0|^2 + |a^1|^2 + |a^2|^2 + a^3|^2]\int d^3x \\ &= [a^* a + (a^* a)^\ddagger]\int d^3x\end{aligned} \tag{8.18}$$

As before, we obtain no restriction on $a = a^\mu e_\mu$ for the positive or negative energy states. The hypercomplex mass has had no effect on this result. There still appear to be twice as many internal states as for the Dirac equation.

The hypernumber a can be written in terms of eigenstates of the spin operator $\frac{1}{2}\hbar e_1$, as follows

$$a = b_{1u}(e_0 + e_1) + b_{2u}(e_2 + ie_3) + b_{1d}(e_2 - ie_3) + b_{2d}(e_0 - e_3) \tag{8.19}$$

We can directly verify that

$$\rho_{\text{rest}} = a^* a + (a^* a)^\ddagger = 2[|b_{1u}|^2 + |b_{2u}|^2 + |b_{1d}|^2 + |b_{2d}|^2] \tag{8.20}$$

The states $(e_0 + e_1)$ and $(e_2 + ie_3)$ are spin-up; the states $(e_0 - e_1)$ and $(e_2 - ie_3)$ are spin-down along x^1, and all four are mutually orthogonal.

For the special case $\psi_v = \psi_a^{\ddagger *}$, we get a reduction in the internal degrees of freedom and an interesting condition on the spin state. Applying $\psi_v \equiv \psi_a^{\ddagger *}$ to equation (8.11) we get

$$\psi_\pm = \begin{pmatrix} L^\ddagger \exp\left[-\frac{i}{\hbar}(\gamma\boldsymbol{\xi}c^2 t - \gamma\boldsymbol{\xi}vx^1)\right]a_\pm \\ L^* \exp\left[-\frac{i}{\hbar}(\gamma\boldsymbol{\xi}c^2 t - \gamma\boldsymbol{\xi}vx^1)\right](a_\pm)^{\ddagger *} \end{pmatrix} \tag{8.21}$$

where

$$a_+^{\ddagger *} = a_+ \qquad \text{and} \qquad a_-^{\ddagger *} = -a_- \tag{8.22}$$

Putting equation (8.22) into equation (8.19) gives

$$a = \pm a^{\ddagger *} \Rightarrow b_{1u}^* = \pm b_{2d} \qquad \text{and} \qquad b_{2u}^* = \mp b_{1d} \tag{8.23}$$

or

$$\begin{aligned} a_+ &= b_{1u}(e_0 + e_1) + b_{1u}^*(e_0 - e_1) + b_{2u}(e_2 + ie_3) - b_{2u}^*(e_2 - ie_3) \\ a_- &= b_{1u}(e_0 + e_1) - b_{1u}^*(e_0 - e_1) + b_{2u}(e_2 + ie_3) + b_{2u}^*(e_2 - ie_3) \end{aligned} \tag{8.24}$$

We are free to set b_{1u} or b_{2u} equal to zero, but in either case the rest state has equal probability of spin-up or spin-down. Remember that e_1 commutes with $\boldsymbol{\xi}$ only when our axes are oriented such that $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are zero. Otherwise, $e_1 \psi$ would not have e_1 operating directly on a.

Thus for particles with $m = -m^\ddagger$, there are only two internal degrees of freedom for each sign of the energy in the rest state, but these are not spin degrees of freedom if e_1 is properly interpreted as the spin operator.

9. *Spin and Spherical Potentials*

If we define the angular momentum operator as

$$\hat{J} \equiv \frac{i}{2}[\mathbf{x}^\ddagger \hat{\mathbf{p}} - \hat{\mathbf{p}}^\ddagger \mathbf{x}] \tag{9.1}$$

where

$$\mathbf{x} = x^k e_k \qquad \text{and} \qquad \mathbf{p} = p^k e_k = i\hbar\, \partial^k e_k = -i\hbar \frac{\partial}{\partial x^k} e_k \tag{9.2}$$

then we obtain

$$\hat{J} = [(x^2 \hat{p}^3 - x^3 \hat{p}^2) + \tfrac{1}{2}\hbar e_1]\, e_1 + (\text{cycl. perm.}) \tag{9.3}$$

Therefore

$$\hat{J}^1 \equiv x^2 \hat{p}^3 - x^3 \hat{p}^2 + \tfrac{1}{2}\hbar e_1 \equiv \hat{L}^1 + \hat{S}^1 \tag{9.4}$$

automatically has the spin operator included in the hyper-number definition of angular momentum.

We define another angular momentum operator

$$\hat{k} \equiv \hat{J} - \tfrac{1}{2}\hbar e_0 = \hat{L}^k e_k + \hbar e_0 \tag{9.5}$$

and find that

$$\hat{k}\hat{k} = (\hat{J}^1)^2 + (\hat{J}^2)^2 + (\hat{J}^3)^2 + \tfrac{1}{4}\hbar^2 \tag{9.6}$$

Thus $\hat{k}$ has eigenvalues (Schiff, 1955) $\pm(j + \frac{1}{2})\hbar = \pm\hbar, \pm 2\hbar, \ldots$, and $\hat{J}$ has eigenvalues $-\frac{1}{2}\hbar, \pm\frac{3}{2}\hbar, \pm\frac{5}{2}\hbar, \ldots$. As we shall see, $\hat{k}$ rather than $\hat{J}$ plays a significant roll in spherically symmetric problems. This is strange since $\hat{J}$ has such a seemingly natural form in the hyper-number formalism. Its lowest eigenvalues do have a peculiar asymmetry, however.

The wave equation, with $i\hbar\partial \to i\hbar\partial - Ae/c$, can be written as

$$i\hbar c\,\partial^0\psi = -c\begin{pmatrix}\hat{\mathbf{p}} & 0\\ 0 & \hat{\mathbf{p}}^\ddagger\end{pmatrix}\psi + \begin{pmatrix}Ae & 0\\ 0 & (Ae)^\ddagger\end{pmatrix}\psi + c^2\begin{pmatrix}0 & m^*\\ m & 0\end{pmatrix}\psi \equiv \hat{H}\psi \tag{9.7}$$

Remember, it is possible that $e' = \mathscr{L}^\ddagger e\mathscr{L}$ and $m' = \mathscr{L}^\ddagger m\mathscr{L}$, which greatly complicates the problem. Perhaps some compelling argument can be found to justify $e \equiv e^\ddagger$ for the charge, as we found for $\hbar \equiv \hbar^\ddagger \equiv \hbar^*$. If $m \neq m^\ddagger$ and/or $e \neq e^\ddagger$, then $\hat{k}$ will seemingly not commute with e or m.

We now convert $\hat{\mathbf{p}}$ to spherical coordinates. By substituting from equation (9.1) and denoting $\mathbf{x} \equiv \mathbf{r}$, we can easily show that

$$\hat{\mathbf{p}} = \frac{\mathbf{r}}{\mathbf{r}^\ddagger\mathbf{r}}[\tfrac{1}{2}(\mathbf{r}^\ddagger\hat{\mathbf{p}} + \hat{\mathbf{p}}^\ddagger\mathbf{r}) - i\hat{J}] \tag{9.8}$$

This gives

$$\hat{\mathbf{p}} = \frac{\mathbf{r}}{\mathbf{r}^\ddagger\mathbf{r}}[\tfrac{3}{2}i\hbar - (x^1\hat{p}^1 + x^2\hat{p}^2 + x^3\hat{p}^3) - i\hat{J}] = -\frac{\mathbf{r}}{r^2}\left[\frac{3}{2}i\hbar + i\hbar r\frac{\partial}{\partial r} - i\hat{J}\right] \tag{9.9}$$

We now define independent spherical hyper-numbers analogous to basis vectors

$$e_r \equiv \frac{x^k e_k}{\sqrt{(\mathbf{xx})}} = \frac{\mathbf{r}}{|\mathbf{r}|}, \qquad e_\theta \equiv \frac{x^1x^3e_1 - x^2x^3e_2 - ((x^1)^2 + (x^1)^2)e_3}{\sqrt{[(x^1)^2 + (x^2)^2]}\,|\mathbf{r}|},$$

$$e_\phi \equiv \frac{-x^2e_1 + x^1e_2}{\sqrt{[(x^1)^2 + (x^2)^2]}} \tag{9.10}$$

It can be directly verified that

$$\begin{gathered} e_re_r = e_0, \quad e_\theta e_\theta = e_0, \quad e_\phi e_\phi = e_0, \quad e_re_\theta = ie_\phi,\\ e_\theta e_\phi = ie_r, \quad e_\phi e_r = ie_\theta, \quad e_re_\theta = -e_\theta e_r, \quad e_\theta e_\phi = -e_\phi e_\theta,\\ e_\phi e_r = -e_re_\phi, \quad e_r^\ddagger = -e_r, \quad e_\theta^\ddagger = -e_\theta \quad \text{and} \quad e_\phi^\ddagger = -e_\phi \end{gathered} \tag{9.11}$$

We then have

$$\hat{\mathbf{p}} = e_r\left(-i\hbar\frac{\partial}{\partial r} - \frac{i}{r}\hbar + \frac{i}{\hbar}\hat{k}\right) \tag{9.12}$$

By using equations (9.1), (9.5) and (9.10) we can show that $\hat{k}$ and e_r anticommute. Also,

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

anticommute; therefore

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{k}$$

commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{p}}$$

That $\hat{k}$ commutes with $1/r$ follows from equation (9.5) and the fact that $\hat{L}^k$ involves only θ and ϕ and their differentials.

If we now try to solve say the coulomb problem we run up against the problem of separating ψ into radial and angular parts. This needs further study. If we make the plausable assumptions that in equations (9.7) and (9.12), $\hat{k}$ can be replaced by its eigenvalue $\hbar k$ if ψ is taken to be $\psi^\mu(r)e_\mu$, and $m = m^\ddagger$, $e = e^\ddagger$, we get the same energy levels as the Dirac equation for the coulomb potential.

This shows that the hyper-number formulation given here is probably correct for ordinary quantum phenomena, but does it contain anything new? The anisotropic hyper-mass and the generalized eigenproblem are, I think, new. Whether they will prove physically significant remains to be seen.

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